

# The Infrared Structure of QCD Amplitudes and $H \rightarrow gg$ in FDH and DRED

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We consider variants of dimensional regularization, including the four-dimensional helicity scheme (FDH) and dimensional reduction (DRED), and present the gluon and quark form factors in the FDH scheme at next-to-next-to-leading order. We also discuss the generalization of the infrared factorization formula to FDH and DRED. This allows us to extract the cusp anomalous dimension as well as the quark and gluon anomalous dimensions at next-to-next-to-leading order in the FDH and DRED scheme, using  $\overline{\text{MS}}$  and  $\overline{\text{DR}}$  renormalization. To obtain these results we also present the renormalization procedure in these schemes.

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## 1 Introduction

The calculation of cross sections beyond leading order in perturbation theory is of utmost importance to fully exploit the wealth of experimental data provided by particle colliders. Computations at next-to-leading order (NLO) are by now standard and can be done in most cases in a fully automated way. At next-to-next-to-leading order (NNLO) the situation is considerably more complicated and only a small number of processes have been computed so far.

Beyond leading order, QCD cross sections are typically split into several parts. At NLO there are virtual and real corrections, at NNLO there are two-loop virtual, virtual-real and double real corrections. Virtual corrections involve the calculation of loop diagrams and only the sum of all contributions leads to finite results.

At intermediate steps of loop calculations ultraviolet (UV) and infrared (IR) divergences need to be regularized. Conventional dimensional regularization (CDR), where all vector bosons are treated in  $D = 4 - 2\epsilon$  dimensions, is not always the optimal choice. Alternatives are the 't Hooft-Veltman scheme (HV) [1], the four-dimensional helicity (FDH) scheme [2], and dimensional reduction (DRED) [3]. In the latter two, vector bosons are treated in 4 dimensions — as far as possible. As an example of the use of the different schemes we mention the two-loop QCD results for the gluon-gluon and quark-gluon scattering. Initially, the interference of these two-loop amplitudes with the tree level was calculated in CDR [4, 5]. Later the helicity amplitudes were computed in the HV and FDH scheme [6, 7]. Clearly a full understanding of the relation between the virtual corrections in the various schemes is required if the FDH or the DRED scheme is to be used for the computation of physical cross sections. Thus, the scheme dependence of UV and IR singularities has to be studied.

The proper treatment of UV singularities of pure QCD amplitudes in the FDH and DRED scheme is well understood. The crucial step is to split quasi-4-dimensional gluons into  $D$ -component gauge fields and  $N_\epsilon = 2\epsilon$  scalar fields, so-called  $\epsilon$ -scalars. During the renormalization process the couplings of the  $\epsilon$ -scalars must be treated as independent, resulting in different renormalization constants and  $\beta$ -functions. Ignoring this distinction can lead to wrong results, violation of unitarity, and the non-cancellation of divergences [8] (see Ref. [9] for potential simplifications and alternative approaches). The independent couplings and their renormalization were already necessities in the equivalence proof of DRED and CDR [10, 11], and in explicit multi-loop calculations in DRED [12–14].

In non-supersymmetric theories the fact that we have different couplings considerably complicates the renormalization procedure. A case of particular interest is the gluon form factor, i.e. the amplitude for the process Higgs to two gluons. This process is described by an effective Higgs-gluon-gluon vertex including the effective coupling  $\lambda$  and has not been calculated at the two-loop level in FDH or DRED so far. In these schemes there is an additional coupling  $\lambda_\epsilon$  between the Higgs and two  $\epsilon$ -scalars and the renormalization becomes highly non-trivial.

The split of gluons was also an essential ingredient in the resolution [15] of the DRED factorization problem [16, 17] and lead to a better understanding of the one-loop transition

rules of Ref. [18]. It is clear that such a split has to be the starting point for a consistent description of IR singularities in the FDH and DRED scheme.

In recent years a lot of progress has been made on the understanding of the IR structure of gauge theories. In Refs. [19–21], a very simple all-order formula predicting the IR divergences of pure QCD amplitudes in CDR has been proposed. An extension of this to the FDH scheme, based on Ref. [19], has been presented by Kilgore [22], where transition rules for NNLO amplitudes computed in the FDH scheme to the CDR (HV) scheme were derived (for recent work on the scheme dependence of double collinear splitting amplitudes see Ref. [23]). At one loop, the corresponding transition rules [18] can easily be realized by simple scheme-dependent  $\gamma^i$  constants for external partons  $i$ . Beyond one loop the transition rules are more involved and require a deeper understanding of IR singularities in loop amplitudes.

The aim of this paper is to deepen the understanding of the IR structure of massless QCD amplitudes in FDH and DRED. We explain the generalization of the IR prediction of Refs. [20, 21] for CDR to the cases of FDH and DRED, highlighting in particular the origin of the differences. As an application and test we calculate the space-like two-loop form factors of quarks and gluons in the FDH and DRED scheme. We describe in detail the necessary UV renormalization procedure in the  $\overline{\text{MS}}$  and  $\overline{\text{DR}}$  renormalization scheme and extract the corresponding two-loop anomalous dimensions.

The structure of the paper is as follows: After reminding the reader of the definitions of the various schemes in Section 2, we present a derivation of how to extend the IR structure systematically to FDH and DRED in Section 3. The prediction of the IR structure is then tested in Section 4, where we present the explicit two-loop results for the quark and gluon form factors in FDH. The renormalization procedure in general, and the additional complications arising from the consistent renormalization of the gluon form factor in the FDH scheme in particular, are discussed in Section 5. With the help of these results, in Section 6 we are able to extract the cusp anomalous dimension as well as the anomalous dimensions of quarks and gluons at NNLO. These results are obtained in the  $\overline{\text{MS}}$  scheme, treating the  $\epsilon$ -scalars as independent particles with multiplicity  $N_\epsilon$ . In Section 7 we then show that the formalism also applies to the  $\overline{\text{DR}}$  renormalization scheme, before presenting our concluding remarks in Section 8.

## 2 QCD in different regularization schemes

In all dimensional regularization schemes, momenta and space-time are continued to  $D = 4 - 2\epsilon$  dimensions. UV and IR divergences of loop and phase space integrals then appear as  $1/\epsilon$  poles. In gauge theories, such as QCD, the vector fields can be treated in different ways. Following the detailed definitions in Ref. [24], we distinguish four schemes: conventional dimensional regularization (CDR), the 't Hooft Veltman scheme (HV), the four-dimensional helicity scheme (FDH), and regularization by dimensional reduction (DRED).

In CDR and HV, gluons are regularized in  $D$  dimensions; the associated  $D$ -dimensional metric tensor is denoted by  $\hat{g}^{\mu\nu}$ . In FDH and DRED, gluons are regularized in 4 dimensions; the associated 4-dimensional metric tensor is denoted by  $g^{\mu\nu}$ . Gauge invariance on the

regularized level requires that a  $D$ -dimensional covariant derivative can be formed. Hence the  $D$ -dimensional space of momenta must be a subspace of the 4-dimensional space of the gluons. The metric tensors thus satisfy the relations

$$g^{\mu\nu} = \hat{g}^{\mu\nu} + \tilde{g}^{\mu\nu}, \quad g^{\mu\nu} \hat{g}_\nu{}^\rho = \hat{g}^{\mu\rho}, \quad g^{\mu\nu} \tilde{g}_\nu{}^\rho = \tilde{g}^{\mu\rho}, \quad \hat{g}^{\mu\nu} \tilde{g}_\nu{}^\rho = 0, \quad (2.1)$$

$$g^{\mu\nu} g_{\mu\nu} = 4, \quad \hat{g}^{\mu\nu} \hat{g}_{\mu\nu} = D, \quad \tilde{g}^{\mu\nu} \tilde{g}_{\mu\nu} = 2\epsilon, \quad (2.2)$$

where a complementary  $2\epsilon$ -dimensional metric  $\tilde{g}^{\mu\nu}$  has been introduced. Mathematical consistency requires [25] that this “4-dimensional” space cannot be the standard Minkowski space, but must be realized as a more complicated space on which these metric tensors can be defined.

Not all gluons need to be regularized, but only internal ones, where “internal” gluons are defined as either virtual gluons that are part of a one-particle irreducible loop diagram or, for real correction diagrams, gluons in the initial or final state that are collinear or soft. “External gluons” are defined as all other gluons. In CDR and DRED, external gluons are treated in the same way as internal ones. In HV and FDH, external gluons are not regularized. In FDH this implies that one needs to distinguish two 4-dimensional spaces—the one of the internal, regularized gluons (metric  $g^{\mu\nu}$ ) and the usual 4-dimensional Minkowski space (metric  $\bar{g}^{\mu\nu}$ ). Table 1 summarizes the definitions of the four regularization schemes.

In FDH and DRED, the (quasi-)4-dimensional regularized gluons can be split into  $D$ -dimensional gluons (which appear in the  $D$ -dimensional covariant derivative as gauge fields) and so-called  $\epsilon$ -scalars with multiplicity  $N_\epsilon = 2\epsilon$  [26]. Often, this split is optional, but as discussed in the introduction in some cases it is essential, see Refs. [8, 10–15]. In QCD with  $N_F$  massless quarks we have to distinguish between

- the gauge coupling  $\alpha_s$ , appearing in all couplings of the  $D$ -dimensional gluons,
- the Yukawa-like evanescent coupling  $\alpha_e$  between  $\epsilon$ -scalars and quarks, and
- the quartic  $\epsilon$ -scalar coupling  $\alpha_{4\epsilon}$ . There are in principle several independent such couplings, differing by the color structure of the respective interactions, but in the present paper this distinction is not necessary.

The renormalization is done by replacing the bare coupling constants with the renormalized ones. Most importantly, all couplings renormalize differently and the  $\beta$ -functions for  $\alpha_s$  and  $\alpha_e$  needed in this paper are given by

$$\mu^2 \frac{d}{d\mu^2} \frac{\alpha_s}{4\pi} = \bar{\beta}(\alpha_s, \alpha_e, \epsilon) = -\epsilon \frac{\alpha_s}{4\pi} - \sum_{m+n}^3 \bar{\beta}_{mn} \left( \frac{\alpha_s}{4\pi} \right)^m \left( \frac{\alpha_e}{4\pi} \right)^n + \mathcal{O}(\alpha^4), \quad (2.3a)$$

$$\mu^2 \frac{d}{d\mu^2} \frac{\alpha_e}{4\pi} = \bar{\beta}^e(\alpha_s, \alpha_e, \epsilon) = -\epsilon \frac{\alpha_e}{4\pi} - \sum_{m+n}^2 \bar{\beta}_{mn}^e \left( \frac{\alpha_s}{4\pi} \right)^m \left( \frac{\alpha_e}{4\pi} \right)^n + \mathcal{O}(\alpha^3). \quad (2.3b)$$

The quartic coupling  $\alpha_{4\epsilon}$  does not appear at this level. Here and in the following the bar denotes quantities obtained using FDH or DRED regularization. In practical calculations,

	CDR	HV	FDH	DRED
internal gluon	$\hat{g}^{\mu\nu}$	$\hat{g}^{\mu\nu}$	$g^{\mu\nu}$	$g^{\mu\nu}$
external gluon	$\hat{g}^{\mu\nu}$	$\bar{g}^{\mu\nu}$	$\bar{g}^{\mu\nu}$	$g^{\mu\nu}$

**Table 1.** Treatment of internal and external gluons in the four different regularization schemes, i.e. prescription which metric tensor has to be used in propagator numerators and polarization sums.

all couplings can be set numerically equal,  $\alpha_s = \alpha_e = \alpha_{4\epsilon}$ , but the  $\beta$ -functions and the related renormalization constants must be treated separately. In contrast to this, in CDR there is just the coupling  $\alpha_s$  and we write the well-known  $\beta$ -function as

$$\mu^2 \frac{d}{d\mu^2} \frac{\alpha_s}{4\pi} = \beta(\alpha_s, \epsilon) = -\epsilon \frac{\alpha_s}{4\pi} - \sum_m^3 \beta_{m0} \left( \frac{\alpha_s}{4\pi} \right)^m + \mathcal{O}(\alpha^4). \quad (2.4)$$

The starting point of the considerations in the next sections are known  $\overline{\text{MS}}$  results of CDR amplitudes. Because of this we use the following renormalization prescription in the FDH and DRED scheme: We treat  $\epsilon$ -scalars as independent scalar particles with an initially arbitrary multiplicity  $N_\epsilon$ . In the  $\overline{\text{MS}}$  scheme we therefore subtract divergences of the form  $\left(\frac{N_\epsilon}{\epsilon}\right)^n$ . As a consequence,  $\bar{\beta}$  and  $\bar{\beta}^e$  depend on the multiplicity  $N_\epsilon$  of the  $\epsilon$ -scalars, and the value of the renormalized coupling  $\alpha_s$  in this scheme equals the corresponding  $\overline{\text{MS}}$  value in CDR.

### 3 Infrared structure

On-shell scattering amplitudes in massless gauge theories remain divergent even after UV renormalization. Fortunately, the remaining infrared divergences factorize in a way that they can be absorbed by a multiplicative renormalization, see Refs. [19–21, 27, 28].

In the following we recapitulate the derivation of the factorization formula in CDR and show how it has to be modified in the cases of FDH and DRED.

#### 3.1 CDR

In the framework of dimensional regularization massless QCD amplitudes with  $n$  external partons can be written in the basis of an adequate color space as

$$\mathcal{M}_n \left( \epsilon, \frac{p_i}{\mu_r}, \alpha_s(\mu_r) \right) = \mathbf{Z} \left( \epsilon, \frac{p_i}{\mu_f}, \alpha_s(\mu_f) \right) H_n \left( \frac{p_i}{\mu_r}, \frac{\mu_f}{\mu_r}, \alpha_s(\mu_r) \right). \quad (3.1)$$

Here,  $H_n$  denotes an arbitrary UV renormalized scattering amplitude, which is finite in the limit  $\epsilon \rightarrow 0$ . Besides the momenta of the external partons,  $p_i$ , and the running strong coupling,  $\alpha_s(\mu_r)$ , it depends explicitly on the renormalization scale,  $\mu_r$ , and the factorization scale,  $\mu_f$ . To simplify things we set  $\mu_r = \mu_f = \mu$  in the following. All soft and collinear divergences of  $\mathcal{M}_n$  are combined in the renormalization factor  $\mathbf{Z}$ .

In minimal subtraction schemes  $\mathbf{Z}$  obeys a renormalization group equation (RGE) with a *finite*,  $\epsilon$ -independent, anomalous dimension,

$$\frac{d}{d \ln \mu} \mathbf{Z} \left( \epsilon, \frac{p_i}{\mu}, \alpha_s(\mu) \right) = -\mathbf{\Gamma} \left( \frac{p_i}{\mu}, \alpha_s(\mu) \right) \mathbf{Z} \left( \epsilon, \frac{p_i}{\mu}, \alpha_s(\mu) \right), \quad (3.2)$$

whose solution is given by the path ordered integral

$$\mathbf{Z} \left( \epsilon, \frac{p_i}{\mu}, \alpha_s(\mu) \right) = -\mathcal{P} \exp \int_0^\mu \frac{d\lambda}{\lambda} \mathbf{\Gamma} \left( \frac{p_i}{\lambda}, \alpha_s(\lambda) \right). \quad (3.3)$$

In Refs. [19–21] arguments are put forward in favor of a conjecture for  $\mathbf{\Gamma}$ , which holds at least up to the two-loop level:

$$\mathbf{\Gamma} \left( \frac{p_i}{\mu}, \alpha_s(\mu) \right) = \sum_{(i,j)}^n \frac{\mathbf{T}_i \cdot \mathbf{T}_j}{2} \gamma^{\text{cusp}}(\alpha_s(\mu)) \ln \frac{\mu^2}{-s_{ij}} + \sum_i \gamma^i(\alpha_s(\mu)). \quad (3.4)$$

The first sum describes the interaction of partons  $i$  and  $j$ . Due to large cancellations beyond the one-loop level only two-particle interactions occur. This term contains the product  $\mathbf{T}_i \cdot \mathbf{T}_j$  of the color generators of partons  $i$  and  $j$ , the kinematic variable  $s_{ij} = \pm 2 p_i \cdot p_j$ , where the negative sign occurs if not all momenta are incoming or outgoing, and the cusp anomalous dimension  $\gamma^{\text{cusp}}$ . The second sum represents the collinear exchange of gluons and is given by the anomalous dimensions  $\gamma^i$  of all external partons  $i$ . In CDR the anomalous dimensions  $\gamma^{\text{cusp}}$  and  $\gamma^i$  are known up to 3-loop order.

A direct consequence of the simple form of Eq. (3.4) is that the commutator  $[\mathbf{\Gamma}(\mu_1), \mathbf{\Gamma}(\mu_2)]$  vanishes and the path ordering in Eq. (3.3) can be neglected. Thus, the determination of  $\mathbf{Z}$  reduces to a simple integration of  $\mathbf{\Gamma}$ . Here, one has to take into account that the scale dependence of  $\mathbf{\Gamma}$  is an explicit and implicit one via the running of  $\alpha_s$ . Because of this one first has to solve the RGE Eq. (2.4) to express  $\alpha_s(\lambda)$  as a power series in  $\alpha_s(\mu)$ , and then integrate Eq. (3.3). At this point it is noteworthy that  $\mathbf{\Gamma}$  itself does not depend explicitly on the regularization parameter  $\epsilon$ . The  $\epsilon$ -poles of  $\mathbf{Z}$  are a direct consequence of these two integrations.

Since the explicit scale dependence in Eq. (3.4) is a logarithmic one it is useful to introduce the partial derivative of  $\mathbf{\Gamma}$

$$\mathbf{\Gamma}'(\alpha_s(\mu)) = \frac{\partial}{\partial \ln \mu} \mathbf{\Gamma} \left( \frac{p_i}{\mu}, \alpha_s(\mu) \right) = -\gamma^{\text{cusp}}(\alpha_s(\mu)) \sum_i C_i. \quad (3.5)$$

Here, the last equality follows from color conservation, e. g.  $\sum_i \mathbf{T}_i \mathcal{M}_n = 0$ , and  $\mathbf{T}_i^2 = C_i$ , where  $C_i = C_{\bar{q}} = C_q = C_F$  for (anti-)quarks and  $C_i = C_g = C_A$  for gluons.

Now we specialize to the case of the space-like quark and gluon form factors, where only two external colored partons appear. Their momenta are normalized to  $s_{12} = +2p_1 \cdot p_2 = -1$  and the expansion in terms of the coupling  $\alpha_s(\mu)$  reduces to

$$\mathbf{\Gamma} \left( \frac{p_i}{\mu}, \alpha_s(\mu) \right) = \sum_{m=1}^{\infty} \left( \frac{\alpha_s}{4\pi} \right)^m \left( \mathbf{\Gamma}'_m \ln \mu + \mathbf{\Gamma}_m \right), \quad (3.6)$$

with

$$\Gamma'_m = -2\gamma_m^{\text{cusp}} C_{q/g}, \quad (3.7a)$$

$$\Gamma_m = +2\gamma_m^{q/g}. \quad (3.7b)$$

On the r.h.s. of Eq. (3.6) and in the following the argument of  $\alpha_s(\mu)$  is suppressed. Finally, Eq. (3.3) yields for the case of form factors

$$\ln \mathbf{Z} = \left(\frac{\alpha_s}{4\pi}\right) \left(\frac{\Gamma'_1}{4\epsilon^2} + \frac{\Gamma_1}{2\epsilon}\right) + \left(\frac{\alpha_s}{4\pi}\right)^2 \left(-\frac{3\beta_{20}\Gamma'_1}{16\epsilon^3} + \frac{\Gamma'_2 - 4\beta_{20}\Gamma_1}{16\epsilon^2} + \frac{\Gamma_2}{4\epsilon}\right) + \mathcal{O}(\alpha_s^3). \quad (3.8)$$

Since  $\ln \mathbf{Z} = \sum_m \left(\frac{\alpha_s}{4\pi}\right)^m (\ln \mathbf{Z})^{(m)}$  absorbs all infrared divergences of  $\mathcal{M}_n$  the following relations for the first coefficients hold:

$$(\ln \mathbf{Z})^{(1)} = \mathcal{M}_n^{(1)} \Big|_{\text{poles}}, \quad (3.9a)$$

$$(\ln \mathbf{Z})^{(2)} = \mathcal{M}_n^{(2)} \Big|_{\text{poles}} - \frac{1}{2} \left(\mathcal{M}_n^{(1)}\right)^2 \Big|_{\text{poles}}. \quad (3.9b)$$

With these formulas it is possible to determine the coefficients of  $\ln \mathbf{Z}$  by a comparison with the IR pole structure of UV renormalized amplitudes.

### 3.2 FDH and DRED

In the FDH and DRED scheme the logic of the derivation is unchanged. The crucial difference is that all quantities depend on the additional couplings  $\alpha_e$  and  $\alpha_{4\epsilon}$ . We stress that although these two couplings are regularization artifacts the behavior is the one of a gauge theory with scalar fields (whose multiplicity happens to be  $N_\epsilon$ ) and with Yukawa-like and quartic scalar interactions.

In the case of the renormalized two-loop quark and gluon form factors the quartic coupling  $\alpha_{4\epsilon}$  does not appear and the divergences can be absorbed by the modified renormalization factor,

$$\bar{\mathbf{Z}} \left( \epsilon, \frac{p_i}{\mu}, \alpha_s(\mu), \alpha_e(\mu) \right) = -\mathcal{P} \exp \int_0^\mu \frac{d\lambda}{\lambda} \bar{\Gamma} \left( \frac{p_i}{\lambda}, \alpha_s(\lambda), \alpha_e(\lambda) \right). \quad (3.10)$$

Likewise, the generalized anomalous dimension  $\bar{\Gamma}$  depends on the couplings  $\alpha_s$  and  $\alpha_e$ :

$$\bar{\Gamma} \left( \frac{p_i}{\mu}, \alpha_s(\mu), \alpha_e(\mu) \right) = \sum_{(i,j)}^n \frac{\mathbf{T}_i \cdot \mathbf{T}_j}{2} \bar{\gamma}^{\text{cusp}}(\alpha_s(\mu), \alpha_e(\mu)) \ln \frac{\mu^2}{-s_{ij}} + \sum_i \bar{\gamma}^i(\alpha_s(\mu), \alpha_e(\mu)). \quad (3.11)$$

Due to this, one has to solve Eqs. (2.3a) and (2.3b) for  $\alpha_s(\lambda)$  and  $\alpha_e(\lambda)$ , respectively, before integrating Eq. (3.10). Specializing again to the case of form factors and expanding the result as a power series in  $\alpha_s$  and  $\alpha_e$  yields

$$\bar{\Gamma} \left( \frac{p_i}{\mu}, \alpha_s(\mu), \alpha_e(\mu) \right) = \sum_{m+n=1}^{\infty} \left(\frac{\alpha_s}{4\pi}\right)^m \left(\frac{\alpha_e}{4\pi}\right)^n \left(\bar{\Gamma}'_{mn} \ln \mu + \bar{\Gamma}_{mn}\right), \quad (3.12)$$

with

$$\bar{\Gamma}'_{mn} = -2\bar{\gamma}_{mn}^{\text{cusp}} C_{q/g}, \quad (3.13a)$$

$$\bar{\Gamma}_{mn} = +2\bar{\gamma}_{mn}^{q/g}. \quad (3.13b)$$

This leads to a modified expression for the renormalization factor,

$$\begin{aligned} \ln \bar{\mathbf{Z}} = & \left(\frac{\alpha_s}{4\pi}\right) \left(\frac{\bar{\Gamma}'_{10}}{4\epsilon^2} + \frac{\bar{\Gamma}_{10}}{2\epsilon}\right) + \left(\frac{\alpha_e}{4\pi}\right) \left(\frac{\bar{\Gamma}'_{01}}{4\epsilon^2} + \frac{\bar{\Gamma}_{01}}{2\epsilon}\right) \\ & + \left(\frac{\alpha_s}{4\pi}\right)^2 \left(-\frac{3\bar{\beta}_{20}\bar{\Gamma}'_{10}}{16\epsilon^3} + \frac{\bar{\Gamma}'_{20} - 4\bar{\beta}_{20}\bar{\Gamma}_{10}}{16\epsilon^2} + \frac{\bar{\Gamma}_{20}}{4\epsilon}\right) \\ & + \left(\frac{\alpha_s}{4\pi}\right) \left(\frac{\alpha_e}{4\pi}\right) \left(-\frac{3\bar{\beta}_{11}^e\bar{\Gamma}'_{01}}{16\epsilon^3} + \frac{\bar{\Gamma}'_{11} - 4\bar{\beta}_{11}^e\bar{\Gamma}_{01}}{16\epsilon^2} + \frac{\bar{\Gamma}_{11}}{4\epsilon}\right) \\ & + \left(\frac{\alpha_e}{4\pi}\right)^2 \left(-\frac{3\bar{\beta}_{02}^e\bar{\Gamma}'_{01}}{16\epsilon^3} + \frac{\bar{\Gamma}'_{02} - 4\bar{\beta}_{02}^e\bar{\Gamma}_{01}}{16\epsilon^2} + \frac{\bar{\Gamma}_{02}}{4\epsilon}\right) + \mathcal{O}(\alpha^3). \end{aligned} \quad (3.14)$$

Comparing this to Eq. (3.8), we notice that the differences between the schemes are considerably more involved than at the one-loop level. Beyond one loop it is not possible any longer to absorb all differences into shifts of the coefficients in Eqs. (3.7a) and (3.7b). The additional terms in Eq. (3.14) depend on the  $\beta$ -function  $\beta_e$  and/or the evanescent coupling  $\alpha_e$  and have a much more complicated structure. However, as expected, in the limit  $\alpha_e \rightarrow 0$ , Eq. (3.14) reduces to the CDR prediction. The appearing  $\beta$ -functions can be taken from the literature, see e.g. Refs. [12, 13, 29–32], and the only free parameters are the anomalous dimensions  $\bar{\Gamma}'_{ij}$  and  $\bar{\Gamma}_{ij}$ . Again, they can be determined by comparing the divergence structure with explicit calculations, see Eqs. (3.9a) and (3.9b). In the next section this is done for the space-like form factors of quarks and gluons.

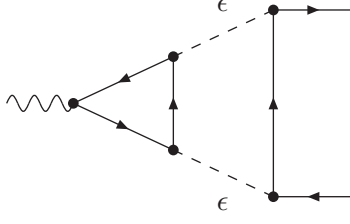
#### 4 Examples: Form factors of quarks and gluons in CDR and FDH

The two-loop results of the quark and gluon form factors in CDR are known for quite some time [33, 34], and in fact even the three-loop results are available [35]. The divergent parts of the three-loop form factors in CDR [36, 37] have been used to extract the anomalous dimensions  $\gamma^q$  [38],  $\gamma^g$  [21], and the cusp anomalous dimension [39] up to three-loop order.

In this section, we present the two-loop results of the quark and gluon form factors obtained from an explicit calculation in the FDH scheme. Since we are not considering contributions from external  $\epsilon$ -scalars, this is equivalent to the DRED scheme. The difference between the CDR and FDH results is due to diagrams with internal  $\epsilon$ -scalars and, therefore, will also involve the couplings  $\alpha_e$  and  $\alpha_{4\epsilon}$ .

To perform the calculations we used the following setup: the generation of the diagrams and the implementation of the Feynman rules is done with the Mathematica package FeynArts [40]; the subsequent evaluation of the algebra in  $D$  and 4 dimensions is then performed with the package TRACER [41]. For the reduction and evaluation of the planar integrals we implemented an in-house algorithm based on integration-by-parts methods and the Laporta-algorithm [42]. The non-planar integrals were reduced and evaluated with the packages FIRE [43] and FIESTA [44], respectively.





**Figure 1.** Two-loop sample diagram resulting in a contribution  $\propto \alpha_e^2$  to the quark form factor.

#### 4.1 Quark form factor

At one loop, the quark form factor in FDH receives additional contributions  $\propto \alpha_e$  from internal  $\epsilon$ -scalars coupling to quarks. Due to the Ward identity, no renormalization is required. The explicit results in CDR and FDH, normalized to tree level, are denoted as  $F$  and  $\bar{F}$ , respectively. They read

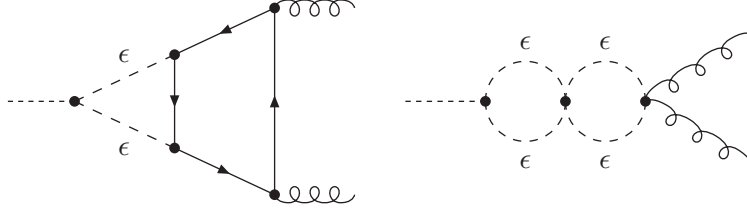
$$F_q^{1l}(\alpha_s) = \left(\frac{\alpha_s}{4\pi}\right) C_F \left[ -\frac{2}{\epsilon^2} - \frac{3}{\epsilon} + \left(-8 + \frac{\pi^2}{6}\right) + \epsilon \left(-16 + \frac{\pi^2}{4} + \frac{14\zeta(3)}{3}\right) + \epsilon^2 \left(-32 + \frac{2\pi^2}{3} + \frac{47\pi^4}{720} + 7\zeta(3)\right) \right] + \mathcal{O}(\epsilon^3), \quad (4.1a)$$

$$\bar{F}_q^{1l}(\alpha_s, \alpha_e) = F_q^{1l}(\alpha_s) + \left(\frac{\alpha_e}{4\pi}\right) N_\epsilon C_F \left[ \frac{1}{2\epsilon} + \frac{1}{2} + \epsilon \left( \frac{1}{2} - \frac{\pi^2}{24} \right) \right] + \mathcal{O}(N_\epsilon \epsilon^2). \quad (4.1b)$$

The additional  $\epsilon$ -scalar contributions in Eq. (4.1b) are proportional to  $\alpha_e$  and  $N_\epsilon$ .

Apart from contributions  $\propto \alpha_s^2$ , the two-loop quark form factor in FDH,  $\bar{F}_q^{2l}(\alpha_s, \alpha_e)$ , contains also terms  $\propto \alpha_s \alpha_e$  and  $\propto \alpha_e^2$ . An example of a diagram contributing to the latter is shown in Figure 1. Performing the explicit calculations in CDR and FDH and forming the expressions relevant for  $\ln \bar{\mathbf{Z}}$  we find

$$\begin{aligned} Q^{(2)}(\alpha_s) &\equiv F_q^{2l}(\alpha_s) - \frac{1}{2} \left( F_q^{1l}(\alpha_s) \right)^2 \\ &= \left( \frac{\alpha_s}{4\pi} \right)^2 \left\{ C_A C_F \left[ \frac{11}{2\epsilon^3} + \frac{\frac{16}{9} + \frac{\pi^2}{6}}{\epsilon^2} - \frac{\frac{961}{108} + \frac{11\pi^2}{12} - 13\zeta(3)}{\epsilon} \right. \right. \\ &\quad \left. \left. - \frac{51157}{648} + \frac{11\pi^4}{45} - \frac{337\pi^2}{108} + \frac{313\zeta(3)}{9} \right] \right. \\ &\quad \left. + C_F^2 \left[ \frac{-\frac{3}{4} + \pi^2 - 12\zeta(3)}{\epsilon} - \frac{1}{8} - \frac{11\pi^4}{45} + \frac{29\pi^2}{6} - 30\zeta(3) \right] \right. \\ &\quad \left. + C_F N_F \left[ -\frac{1}{\epsilon^3} - \frac{4}{9\epsilon^2} + \frac{\frac{65}{54} + \frac{\pi^2}{6}}{\epsilon} + \frac{4085}{324} + \frac{23\pi^2}{54} + \frac{2\zeta(3)}{9} \right] \right\} \\ &\quad + \mathcal{O}(\epsilon^1), \end{aligned} \quad (4.2a)$$



**Figure 2.** Sample diagram contributing to the gluon form factor  $\propto \lambda_\epsilon \alpha_e \alpha_s$  and  $\lambda_\epsilon \alpha_{4\epsilon} \alpha_s$ , respectively.

$$\begin{aligned}
\bar{Q}^{(2)}(\alpha_s, \alpha_e) &\equiv \bar{F}_q^{2l}(\alpha_s, \alpha_e) - \frac{1}{2} \left( \bar{F}_q^{1l}(\alpha_s, \alpha_e) \right)^2 \\
&= Q^{(2)}(\alpha_s) + \left( \frac{\alpha_s}{4\pi} \right)^2 N_\epsilon \left\{ C_A C_F \left[ -\frac{1}{4\epsilon^3} - \frac{1}{36\epsilon^2} + \frac{\frac{167}{216} + \frac{\pi^2}{24}}{\epsilon} \right] \right\} \\
&\quad + \left( \frac{\alpha_s}{4\pi} \right) \left( \frac{\alpha_e}{4\pi} \right) N_\epsilon \left\{ C_A C_F \frac{11}{4\epsilon} + C_F^2 \left[ -\frac{3}{2\epsilon^2} - \frac{1 + \frac{\pi^2}{6}}{\epsilon} \right] \right\} \\
&\quad + \left( \frac{\alpha_e}{4\pi} \right)^2 N_\epsilon \left\{ C_A C_F \left[ \frac{-\frac{1}{2} + \frac{N_\epsilon}{4}}{\epsilon^2} \right] + C_F^2 \left[ \frac{1}{\epsilon^2} - N_\epsilon \left( \frac{1}{4\epsilon^2} + \frac{1}{16\epsilon} \right) \right] \right. \\
&\quad \left. + C_F N_F \left[ \frac{1}{4\epsilon^2} - \frac{3}{8\epsilon} \right] \right\} + \mathcal{O}(N_\epsilon \epsilon^0).
\end{aligned} \tag{4.2b}$$

Again, all additional terms in the FDH result (4.2b) are proportional to at least one power of  $N_\epsilon$ ; in the contributions proportional to  $\alpha_e^2$  even  $N_\epsilon^2$  terms occur. All results have been obtained using  $\overline{\text{MS}}$  renormalization of  $\alpha_s$  and  $\alpha_e$ , Eqs. (2.3a) and (2.3b). The renormalization factors are listed in Section 5 for convenience.

## 4.2 Gluon form factor

The form factor of the gluon is computed in an effective theory approach where the coupling  $\lambda$  of the gluon to the Higgs is induced through a dimension 5 operator. The renormalization of this coupling in CDR is well understood [45]. In FDH, the presence of  $\epsilon$ -scalars induces an additional coupling to the Higgs,  $\lambda_\epsilon$ . This coupling is independent of  $\lambda$  and renormalizes differently. In fact the renormalization of  $\lambda$  is also affected by the presence of  $\lambda_\epsilon$ . In Section 5 we explain how to renormalize the gluon form factor in the FDH scheme.

After renormalization, the explicit results for  $F_g^{1l}(\alpha_s)$  and  $\bar{F}_g^{1l}(\alpha_s, \lambda_\epsilon/\lambda)$ , the one-loop gluon form factors in CDR and FDH, respectively, normalized to tree level, read

$$\begin{aligned}
F_g^{1l}(\alpha_s) &= \left( \frac{\alpha_s}{4\pi} \right) \left\{ C_A \left[ -\frac{2}{\epsilon^2} - \frac{11}{3\epsilon} + \frac{\pi^2}{6} + \epsilon \left( -2 + \frac{14\zeta(3)}{3} \right) \right. \right. \\
&\quad \left. \left. + \epsilon^2 \left( -6 + \frac{47\pi^4}{720} \right) \right] + N_F \left( \frac{2}{3\epsilon} \right) \right\} + \mathcal{O}(\epsilon^3),
\end{aligned} \tag{4.3a}$$

$$\bar{F}_g^{1l}(\alpha_s, \lambda_\epsilon/\lambda) = F_g^{1l}(\alpha_s) + \left(\frac{\alpha_s}{4\pi}\right) N_\epsilon C_A \left\{ \frac{1}{6\epsilon} + \frac{\lambda_\epsilon}{\lambda} (1 + 3\epsilon) \right\} + \mathcal{O}(N_\epsilon \epsilon^2). \quad (4.3b)$$

All  $\epsilon$ -scalar terms in the FDH result are proportional to  $\alpha_s$  and  $N_\epsilon$ . The terms proportional to  $\lambda_\epsilon/\lambda$  appear from the ratio of the one-loop diagrams  $\propto \lambda_\epsilon$ , normalized to tree level.

At two loops, the gluon form factor in FDH contains also contributions  $\propto \lambda_\epsilon$ , with some examples shown in Figure 2. However, after renormalization and forming the relevant expressions for  $\ln \bar{\mathbf{Z}}$  the contributions proportional to these couplings drop out, in agreement with the IR prediction (3.14), which cannot contain the coupling  $\lambda_\epsilon$ . The explicit results read

$$\begin{aligned} G^{(2)}(\alpha_s) &\equiv F_g^{2l}(\alpha_s) - \frac{1}{2} \left( F_g^{1l}(\alpha_s) \right)^2 \\ &= \left( \frac{\alpha_s}{4\pi} \right)^2 \left\{ C_A^2 \left[ \frac{11}{2\epsilon^3} + \frac{3 + \frac{\pi^2}{6}}{\epsilon^2} + \frac{-\frac{346}{27} + \frac{11\pi^2}{36} + \zeta(3)}{\epsilon} + \frac{5105}{162} + \frac{67\pi^2}{36} - \frac{143\zeta(3)}{9} \right] \right. \\ &\quad + C_A N_F \left[ -\frac{1}{\epsilon^3} - \frac{17}{9\epsilon^2} + \frac{\frac{64}{27} - \frac{\pi^2}{18}}{\epsilon} - \frac{916}{81} - \frac{5\pi^2}{18} - \frac{46\zeta(3)}{9} \right] \\ &\quad \left. + C_F N_F \left[ \frac{1}{\epsilon} - \frac{67}{6} + 8\zeta(3) \right] + N_F^2 \frac{2}{9\epsilon^2} \right\} + \mathcal{O}(\epsilon^1), \end{aligned} \quad (4.4a)$$

$$\begin{aligned} \bar{G}^{(2)}(\alpha_s, \alpha_e) &\equiv \bar{F}_g^{2l}(\alpha_s, \alpha_e, \lambda_\epsilon/\lambda) - \frac{1}{2} \left( \bar{F}_g^{1l}(\alpha_s, \lambda_\epsilon/\lambda) \right)^2 \\ &= G^{(2)}(\alpha_s) + \left( \frac{\alpha_s}{4\pi} \right)^2 N_\epsilon \left\{ C_A^2 \left[ -\frac{1}{4\epsilon^3} + \frac{-\frac{7}{18} + \frac{N_\epsilon}{72}}{\epsilon^2} + \frac{\frac{49}{27} - \frac{\pi^2}{72}}{\epsilon} \right] + C_A N_F \frac{1}{9\epsilon^2} \right\} \\ &\quad + \left( \frac{\alpha_s}{4\pi} \right) \left( \frac{\alpha_e}{4\pi} \right) N_\epsilon \left\{ -\frac{C_F N_F}{2\epsilon} \right\} + \mathcal{O}(N_\epsilon \epsilon^0). \end{aligned} \quad (4.4b)$$

In contrast to the quark form factor, Eq. (4.2b), the  $\epsilon$ -scalar terms in Eq. (4.4b) are much simpler and do not depend on  $\alpha_e^2$ .

## 5 UV renormalization of the quark and gluon form factor in FDH

Renormalization in the FDH and DRED scheme is considerably more involved than in CDR due to the additional evanescent couplings. Here we present details on the renormalization in these schemes, particularly for the gluon form factor, which involves not only the renormalization of  $\alpha_s$  and  $\alpha_e$  but also of composite operators and the associated couplings  $\lambda$  and  $\lambda_\epsilon$ .

In general, the renormalization of the quark and gluon form factors is done by replacing the bare coupling constants with the renormalized ones,

$$c_{\text{bare}} = c \left( 1 + \sum_i \delta Z_c^{(i)} \right), \quad (5.1)$$

where  $i$  indicates the loop order and  $c \in \{\alpha_s, \lambda\}$  in the case of CDR and  $c \in \{\alpha_s, \alpha_e, \alpha_{4\epsilon}, \lambda, \lambda_\epsilon\}$  in FDH. As always, we use a bar to distinguish quantities in the FDH scheme from corresponding quantities in CDR.

This leads to the following expressions for the coefficients of the renormalized quark form factor in CDR:

$$F_q^{1l}(\alpha_s) = F_{q,\text{bare}}^{1l}(\alpha_s), \quad (5.2a)$$

$$F_q^{2l}(\alpha_s) = F_{q,\text{bare}}^{2l}(\alpha_s) + \delta Z_{\alpha_s}^{(1)} F_{q,\text{bare}}^{1l}(\alpha_s). \quad (5.2b)$$

Due to the QED Ward-identity the photon coupling does not have to be renormalized, and the bare and renormalized form factors are the same at the one-loop level; at the two-loop level only the subloop renormalization of  $\alpha_s$  is necessary.

In FDH, again no renormalization is needed at the one-loop level; at the two-loop level the subloop renormalization of the couplings appearing in the one-loop diagrams is necessary. Since all additional  $\epsilon$ -scalar one-loop diagrams are proportional to  $\alpha_e$ , we can write the FDH renormalization as

$$\bar{F}_q^{1l}(\alpha_s, \alpha_e) = \bar{F}_{q,\text{bare}}^{1l}(\alpha_s, \alpha_e), \quad (5.3a)$$

$$\begin{aligned} \bar{F}_q^{2l}(\alpha_s, \alpha_e) &= \bar{F}_{q,\text{bare}}^{2l}(\alpha_s, \alpha_e) + \delta \bar{Z}_{\alpha_s}^{(1)} \bar{F}_{q,\text{bare}}^{1l}(\alpha_s) \\ &\quad + \delta \bar{Z}_{\alpha_e}^{(1)} \left( \bar{F}_{q,\text{bare}}^{1l}(\alpha_s, \alpha_e) - F_{q,\text{bare}}^{1l}(\alpha_s) \right). \end{aligned} \quad (5.3b)$$

Now we turn to the more complicated case of the gluon form factor. Already at tree level it is proportional to the coupling  $\lambda$ , which needs to be renormalized. Besides, the subloop renormalization of both couplings appearing in the one-loop diagrams appears at higher orders. Thus, the CDR coefficients of the renormalized gluon form factor, normalized to tree level, read

$$F_g^{1l}(\alpha_s) = F_{g,\text{bare}}^{1l}(\alpha_s) + \delta Z_\lambda^{(1)}, \quad (5.4a)$$

$$F_g^{2l}(\alpha_s) = F_{g,\text{bare}}^{2l}(\alpha_s) + \left( \delta Z_{\alpha_s}^{(1)} + \delta Z_\lambda^{(1)} \right) F_{g,\text{bare}}^{1l}(\alpha_s) + \delta Z_\lambda^{(2)}. \quad (5.4b)$$

Renormalization in FDH is more complicated because of the additional coupling  $\lambda_\epsilon$  appearing in one-loop diagrams. Since the entire one-loop difference between FDH and CDR is  $\propto \lambda_\epsilon \alpha_s$ , we can write

$$\bar{F}_g^{1l}(\alpha_s, \lambda_\epsilon/\lambda) = \bar{F}_{g,\text{bare}}^{1l}(\alpha_s, \lambda_\epsilon/\lambda) + \delta \bar{Z}_\lambda^{(1)}, \quad (5.5a)$$

$$\begin{aligned} \bar{F}_g^{2l}(\alpha_s, \alpha_e, \lambda_\epsilon/\lambda) &= \bar{F}_{g,\text{bare}}^{2l}(\alpha_s, \alpha_e, \alpha_{4\epsilon}, \lambda_\epsilon/\lambda) + \left( \delta \bar{Z}_{\alpha_s}^{(1)} + \delta \bar{Z}_\lambda^{(1)} \right) \bar{F}_{g,\text{bare}}^{1l}(\alpha_s) \\ &\quad + \left( \delta \bar{Z}_{\alpha_s}^{(1)} + \delta \bar{Z}_{\lambda_\epsilon}^{(1)} \right) \left( \bar{F}_{g,\text{bare}}^{1l}(\alpha_s, \lambda_\epsilon/\lambda) - F_{g,\text{bare}}^{1l}(\alpha_s) \right) + \delta \bar{Z}_\lambda^{(2)}. \end{aligned} \quad (5.5b)$$

The couplings  $\alpha_e$  and  $\alpha_{4\epsilon}$  only appear in two-loop diagrams and don't have to be renormalized at this level.

The previous equations show which renormalization constants are needed up to which order. In CDR, the required renormalization constants read [45–47]

$$\delta Z_{\alpha_s}^{(1)} = \left( \frac{\alpha_s}{4\pi} \right) \left( -\frac{\beta_{20}}{\epsilon} \right), \quad (5.6a)$$

$$\delta Z_{\alpha_s}^{(2)} = \left( \frac{\alpha_s}{4\pi} \right)^2 \left( \frac{\beta_{20}^2}{\epsilon^2} - \frac{\beta_{30}}{2\epsilon} \right), \quad (5.6b)$$

$$\delta Z_\lambda^{(1)} = \delta Z_{\alpha_s}^{(1)}, \quad (5.6c)$$

$$\delta Z_\lambda^{(2)} = \left(\frac{\alpha_s}{4\pi}\right)^2 \left(\frac{\beta_{20}^2}{\epsilon^2} - \frac{\beta_{30}}{\epsilon}\right). \quad (5.6d)$$

Thus, the whole renormalization of the form factors is described by the  $\beta$ -function of  $\alpha_s$ , defined in Eq. (2.4), whose first non-vanishing coefficients in the  $\overline{\text{MS}}$  scheme are given by [12, 13]

$$\beta_{20} = \frac{11}{3}C_A - \frac{2}{3}N_F, \quad (5.7a)$$

$$\beta_{30} = \frac{34}{3}C_A^2 - \frac{10}{3}C_A N_F - 2C_F N_F. \quad (5.7b)$$

In the FDH scheme, the additional  $\epsilon$ -scalar with multiplicity  $N_\epsilon$  leads to a modification of the renormalization constants for  $\alpha_s$  and  $\lambda$  and to new renormalization constants for  $\alpha_e$  and  $\lambda_\epsilon$ . The necessary FDH renormalization constants in the  $\overline{\text{MS}}$  scheme described in Section 2 read

$$\delta \bar{Z}_{\alpha_s}^{(1)} = \left(\frac{\alpha_s}{4\pi}\right) \left(-\frac{\bar{\beta}_{20}}{\epsilon}\right), \quad (5.8a)$$

$$\delta \bar{Z}_{\alpha_s}^{(2)} = \left(\frac{\alpha_s}{4\pi}\right)^2 \left(\frac{\bar{\beta}_{20}^2}{\epsilon^2} - \frac{\bar{\beta}_{30}}{2\epsilon}\right) + \left(\frac{\alpha_s}{4\pi}\right) \left(\frac{\alpha_e}{4\pi}\right) \left(-\frac{\bar{\beta}_{21}}{2\epsilon}\right), \quad (5.8b)$$

$$\delta \bar{Z}_\lambda^{(1)} = \delta \bar{Z}_{\alpha_s}^{(1)}, \quad (5.8c)$$

$$\delta \bar{Z}_\lambda^{(2)} = \left(\frac{\alpha_s}{4\pi}\right)^2 \left(\frac{\bar{\beta}_{20}^2}{\epsilon^2} - \frac{\bar{\beta}_{30}}{\epsilon}\right) + \left(\frac{\alpha_s}{4\pi}\right) \left(\frac{\alpha_e}{4\pi}\right) \left(1 + \frac{\lambda_\epsilon}{\lambda}\right) \left(-\frac{\bar{\beta}_{21}}{2\epsilon}\right), \quad (5.8d)$$

$$\delta \bar{Z}_{\alpha_e}^{(1)} = \left(\frac{\alpha_s}{4\pi}\right) \left(-\frac{\bar{\beta}_{11}^e}{\epsilon}\right) + \left(\frac{\alpha_e}{4\pi}\right) \left(-\frac{\bar{\beta}_{02}^e}{\epsilon}\right), \quad (5.8e)$$

$$\delta \bar{Z}_{\lambda_\epsilon}^{(1)} = \left(\frac{\alpha_s}{4\pi}\right) \left(-\frac{3C_A}{\epsilon}\right) + \left(\frac{\alpha_e}{4\pi}\right) \frac{N_F}{\epsilon} + \left(\frac{\alpha_{4\epsilon}}{4\pi}\right) C_A \left(\frac{-1 + N_\epsilon}{\epsilon}\right), \quad (5.8f)$$

with the following non-vanishing coefficients of the  $\beta$ -functions defined in Eqs. (2.3a) and (2.3b):

$$\bar{\beta}_{20} = \beta_{20} + N_\epsilon \left(-\frac{C_A}{6}\right), \quad (5.9a)$$

$$\bar{\beta}_{30} = \beta_{30} + N_\epsilon \left(-\frac{7}{3}C_A^2\right), \quad (5.9b)$$

$$\bar{\beta}_{21} = N_\epsilon C_F N_F, \quad (5.9c)$$

$$\bar{\beta}_{11}^e = 6 C_F, \quad (5.9d)$$

$$\bar{\beta}_{02}^e = -4 C_F + 2 C_A - N_F + N_\epsilon (C_F - C_A). \quad (5.9e)$$

The modifications of the  $\alpha_s$  and  $\lambda$  renormalization constants are of the order  $N_\epsilon$  and depends on all couplings including  $\alpha_e$  and  $\lambda_\epsilon$ . The renormalization constant  $\delta \bar{Z}_{\lambda_\epsilon}^{(1)}$  even depends on  $\alpha_{4\epsilon}$ .

The renormalization of  $\alpha_s$  and  $\alpha_e$ , Eqs. (5.8a), (5.8b) and (5.8e), and all appearing  $\beta$ -functions are obtained from Refs. [29–32], where renormalization group equations for general gauge theories are given. We use the  $\overline{\text{MS}}$  renormalization scheme described at the end of Section 2. Extending the formalism described in Ref. [45] yields the renormalization of  $\lambda$ , Eqs. (5.8c) and (5.8d), including the appearance of  $\lambda_e$ . The renormalization of this coupling, Eq. (5.8f), was obtained from an explicit one-loop calculation.

## 6 Results: Anomalous dimensions in FDH and DRED

With the results from Section 4 and Eqs. (3.9a), (3.9b), (3.13a), (3.13b) and (3.14) we are able to extract the scheme dependence of the anomalous dimensions  $\gamma^{\text{cusp}}$ ,  $\gamma^q$  and  $\gamma^g$ . Here, the cusp anomalous  $\gamma^{\text{cusp}}$  can be extracted from both form factors, which allows for a cross check of the method and the explicit calculation.

In the case of CDR we recover the well-known results, see e. g. Ref. [35]

$$\gamma_{10}^{\text{cusp}} = 4, \quad (6.1a)$$

$$\gamma_{20}^{\text{cusp}} = C_A \left( \frac{268}{9} - \frac{4}{3}\pi^2 \right) - \frac{40}{9}N_F, \quad (6.1b)$$

$$\gamma_{10}^q = -3C_F, \quad (6.1c)$$

$$\gamma_{20}^q = C_A C_F \left( -\frac{961}{54} - \frac{11}{6}\pi^2 + 26\zeta(3) \right) + C_F^2 \left( -\frac{3}{2} + 2\pi^2 - 24\zeta(3) \right) \quad (6.1d)$$

$$+ C_F N_F \left( \frac{65}{27} + \frac{\pi^2}{3} \right), \quad (6.1e)$$

$$\gamma_{10}^g = -\beta_{20} = -\frac{11}{3}C_A + \frac{2}{3}N_F, \quad (6.1f)$$

$$\gamma_{20}^g = C_A^2 \left( -\frac{692}{27} + \frac{11}{18}\pi^2 + 2\zeta(3) \right) + C_A N_F \left( \frac{128}{27} - \frac{\pi^2}{9} \right) + 2C_F N_F \quad (6.1g)$$

The additional contributions originating from internal  $\epsilon$ -scalars in the FDH or DRED scheme lead to the following modified anomalous dimensions:

$$\bar{\gamma}_{10}^{\text{cusp}} = \gamma_{10}^{\text{cusp}}, \quad (6.2a)$$

$$\bar{\gamma}_{01}^{\text{cusp}} = 0, \quad (6.2b)$$

$$\bar{\gamma}_{20}^{\text{cusp}} = \gamma_{20}^{\text{cusp}} - N_\epsilon \frac{16}{9}C_A, \quad (6.2c)$$

$$\bar{\gamma}_{11}^{\text{cusp}} = 0, \quad (6.2d)$$

$$\bar{\gamma}_{02}^{\text{cusp}} = 0, \quad (6.2e)$$

$$\bar{\gamma}_{10}^q = \gamma_{10}^q, \quad \bar{\gamma}_{10}^g = \gamma_{10}^g + N_\epsilon \frac{C_A}{6}, \quad (6.2f)$$

$$\bar{\gamma}_{01}^q = N_\epsilon \frac{C_F}{2}, \quad \bar{\gamma}_{01}^g = 0, \quad (6.2g)$$

$$\bar{\gamma}_{20}^q = \gamma_{20}^q + N_\epsilon \left( \frac{167}{108} + \frac{\pi^2}{12} \right) C_A C_F, \quad \bar{\gamma}_{20}^g = \gamma_{20}^g + N_\epsilon \left( \frac{98}{27} - \frac{\pi^2}{36} \right) C_A^2, \quad (6.2h)$$

$$\bar{\gamma}_{11}^q = N_\epsilon \left[ \frac{11}{2} C_A C_F - \left( 2 + \frac{\pi^2}{3} \right) C_F^2 \right], \quad \bar{\gamma}_{11}^g = -\bar{\beta}_{21} = -N_\epsilon C_F N_F, \quad (6.2i)$$

$$\bar{\gamma}_{02}^q = -N_\epsilon \frac{3}{4} C_F N_F - N_\epsilon^2 \frac{C_F^2}{8}, \quad \bar{\gamma}_{02}^g = 0. \quad (6.2j)$$

Generally, all these scheme differences are of  $\mathcal{O}(N_\epsilon)$  or  $\mathcal{O}(N_\epsilon^2)$ , so setting  $N_\epsilon$  to zero in Eqs. (6.2a)–(6.2j) yields the known CDR anomalous dimensions. The one-loop cusp anomalous dimension obtained for both form factors is scheme independent, while at two-loop order there is an additional term in  $\bar{\gamma}_{20}^{\text{cusp}}$ , i.e. a term proportional to  $\alpha_s^2 N_\epsilon$ . The one-loop quark anomalous dimension gets an additional contribution proportional to  $\alpha_e N_\epsilon$ , in the coefficient  $\bar{\gamma}_{01}^q$ , while the  $\alpha_s$  term is unchanged; at two-loop order, all coefficients  $\bar{\gamma}_{mn}^q$  get additional terms. In the  $\alpha_e^2$  part there is even a  $N_\epsilon^2$  term. In the case of gluons the scheme dependence is absorbed by a term  $\propto \alpha_s N_\epsilon$  at the one-loop level, and by terms  $\propto \alpha_s^2 N_\epsilon$  and  $\propto \alpha_s \alpha_e N_\epsilon$  at the two-loop level (there are no terms  $\propto \alpha_e^2$  and no terms containing  $\alpha_{4e}$ ).

Our results can be compared with Ref. [22], where the gluon anomalous dimension has been obtained from the process  $q\bar{q} \rightarrow g\gamma$ . As we are consistently using the  $\overline{\text{MS}}$  scheme, as described at the end of Section 2, the anomalous dimensions given here do not contain terms of  $\mathcal{O}(\epsilon)$ . In Ref. [22], such  $\mathcal{O}(\epsilon)$  terms are included to absorb process-specific contributions and lead to finite differences for the two-loop anomalous dimensions. Further, the  $\mathcal{O}(N_\epsilon^2)$  is missing in Ref. [22], which however plays no role in the factorization formula (3.14) for  $N_\epsilon = 2\epsilon$ .

## 7 Factorization in the $\overline{\text{DR}}$ scheme

Up to now we considered a minimal coupling renormalization where all additional UV singular contributions arising from internal  $\epsilon$ -scalars are removed, including terms of the form  $(\frac{N_\epsilon}{\epsilon})^n$ . Now we show that the IR structure can be described by Eq. (3.14) even if the  $\overline{\text{DR}}$  renormalization scheme is used.

The  $\overline{\text{DR}}$  scheme corresponds to setting  $N_\epsilon = 2\epsilon$  and then subtracting only the remaining  $\frac{1}{\epsilon}$  UV poles. The difference between the  $\overline{\text{MS}}$  and  $\overline{\text{DR}}$  scheme are  $N_\epsilon$  terms in the  $\beta$ -functions and renormalization constants.

As it turns out, the structure of the factorization formula (3.14) is such, that an arbitrary  $N_\epsilon$  term in a  $\beta$  coefficient at the order  $\mathcal{O}(\epsilon^{-n})$  can, for  $N_\epsilon = 2\epsilon$ , be absorbed by a *finite* shift in the anomalous dimensions  $\Gamma'_{mn}$  and  $\Gamma_{mn}$  at the order  $\mathcal{O}(\epsilon^{-n+1})$ . Since in Eq. (3.14) no  $\beta$ -coefficients enter at the one-loop level, all corresponding one-loop anomalous dimensions remain unchanged. Comparing the two-loop form factors renormalized in the  $\overline{\text{DR}}$  scheme with the factorization formula we extract the two-loop anomalous dimension in the

$\overline{\text{DR}}$  scheme and find the following results:

$$\bar{\gamma}_{10}^{\text{cusp},\overline{\text{DR}}} = \gamma_{10}^{\text{cusp}}, \quad (7.1a)$$

$$\bar{\gamma}_{01}^{\text{cusp},\overline{\text{DR}}} = 0, \quad (7.1b)$$

$$\bar{\gamma}_{20}^{\text{cusp},\overline{\text{DR}}} = \gamma_{20}^{\text{cusp}} - \frac{4}{3} C_A, \quad (7.1c)$$

$$\bar{\gamma}_{11}^{\text{cusp},\overline{\text{DR}}} = 0, \quad (7.1d)$$

$$\bar{\gamma}_{02}^{\text{cusp},\overline{\text{DR}}} = 0, \quad (7.1e)$$

$$\bar{\gamma}_{10}^{q,\overline{\text{DR}}} = \gamma_{10}^q, \quad \bar{\gamma}_{10}^{g,\overline{\text{DR}}} = \gamma_{10}^g, \quad (7.1f)$$

$$\bar{\gamma}_{01}^{q,\overline{\text{DR}}} = 0, \quad \bar{\gamma}_{01}^{g,\overline{\text{DR}}} = 0, \quad (7.1g)$$

$$\bar{\gamma}_{20}^{q,\overline{\text{DR}}} = \gamma_{20}^q + \frac{17}{9} C_A C_F, \quad \bar{\gamma}_{20}^{g,\overline{\text{DR}}} = \gamma_{20}^g + \frac{8}{9} C_A^2, \quad (7.1h)$$

$$\bar{\gamma}_{11}^{q,\overline{\text{DR}}} = -\bar{\beta}_{11}^{e,\overline{\text{DR}}} C_F, \quad \bar{\gamma}_{11}^{g,\overline{\text{DR}}} = 0, \quad (7.1i)$$

$$\bar{\gamma}_{02}^{q,\overline{\text{DR}}} = -\bar{\beta}_{02}^{e,\overline{\text{DR}}} C_F, \quad \bar{\gamma}_{02}^{g,\overline{\text{DR}}} = 0, \quad (7.1j)$$

including the non-vanishing  $\beta$ -coefficients

$$\bar{\beta}_{11}^{e,\overline{\text{DR}}} = \bar{\beta}_{11}^e|_{N_\epsilon=0} = 6 C_F, \quad (7.2a)$$

$$\bar{\beta}_{02}^{e,\overline{\text{DR}}} = \bar{\beta}_{02}^e|_{N_\epsilon=0} = -4 C_F + 2 C_A - N_F. \quad (7.2b)$$

As expected, all one-loop quantities coincide with the corresponding  $\overline{\text{MS}}$  values in CDR and the two-loop anomalous dimensions  $\propto \alpha_s^2$  receive finite shifts. Additionally, coefficients of the  $\beta$ -function  $\bar{\beta}^{e,\overline{\text{DR}}}$  appear in the case of the two-loop quark form factor. They are obtained from the previously used  $\overline{\text{MS}}$  coefficients of  $\bar{\beta}^e$  in the limit  $N_\epsilon = 0$ .

While in the case of the previous sections, the shifts in the anomalous dimensions were of the order  $\mathcal{O}(N_\epsilon)$ , the  $\gamma$ -coefficients corresponding to  $\overline{\text{DR}}$  renormalization differ by finite shifts, which do not vanish for  $\epsilon \rightarrow 0$ . This reflects the general fact that anomalous dimensions are renormalization-scheme dependent.

## 8 Conclusion

In this paper we extended the well-known CDR conjecture [20, 21] for the infrared structure of massless QCD amplitudes to the cases of FDH and DRED, see Eq. (3.14). Consistently using the  $\overline{\text{MS}}$  scheme, we extracted the NNLO anomalous dimensions by comparing this conjecture with the form factors of quarks and gluons. In the case of the gluon form factor we explained the necessary renormalization of the effective Higgs couplings  $\lambda$  and  $\lambda_\epsilon$ .

In the  $\overline{\text{MS}}$  scheme we treat the multiplicity  $N_\epsilon$  of the  $\epsilon$ -scalars as an arbitrary quantity that enters in loop diagrams, and the UV renormalization is done by subtracting all



divergent parts, including terms of the form  $\left(\frac{N_\epsilon}{\epsilon}\right)^n$ . The resulting regularization dependence can be absorbed in the modified infrared factorization formula by unambiguously fixed shifts in the anomalous dimensions that are proportional to at least one power of  $N_\epsilon$ , Eqs. (6.2a)–(6.2j). Thus, after renormalization and after subtracting the corresponding IR divergent terms, the difference between an amplitude computed either in CDR or FDH is of the order  $\mathcal{O}(N_\epsilon)$  and free of  $\frac{1}{\epsilon}$ -poles.

This implies that the subtracted results in CDR and FDH are the same for  $N_\epsilon \rightarrow 0$  and it is possible to convert the results between the schemes. The transition rules between FDH and CDR that follow from the anomalous dimensions given in Eqs. (6.1a)–(6.2j) are consistent with the transition rules given by Kilgore [22].

Further we show how the FDH and DRED factorization works in other renormalization schemes, namely in the  $\overline{\text{DR}}$  scheme. Here, only remaining divergences after setting  $N_\epsilon = 2\epsilon$  are subtracted. The resulting regularization dependence is absorbed by finite shifts in the anomalous dimensions that do not depend on  $\epsilon$  or  $N_\epsilon$ , Eqs. (7.1a)–(7.1j). Thus, a transition to the CDR anomalous dimensions like in the  $\overline{\text{MS}}$  case is not possible.

In both renormalization schemes the cusp anomalous dimension extracted from the quark form factor agrees with the corresponding expression obtained from the gluon form factor. This is further evidence for the universality of the proposed infrared structure in the FDH and DRED scheme.

In order to obtain transition rules for two-loop amplitudes in the DRED scheme, processes with external  $\epsilon$ -scalars need to be considered. In particular, the corresponding anomalous dimension has to be computed. This can be done for example by computing the  $\epsilon$ -scalar form factor corresponding to the process Higgs  $\rightarrow$  two  $\epsilon$ -scalars. The investigation of alternative possibilities to compute the anomalous dimensions more directly as well as the application of the transition rules to the results for the  $2 \rightarrow 2$  scattering amplitudes in massless QCD [4–7] is left for future work.

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## References

- [1] G. 't Hooft and M. Veltman, *Regularization and Renormalization of Gauge Fields*, *Nucl.Phys.* **B44** (1972) 189–213.
- [2] Z. Bern and D. A. Kosower, *The Computation of loop amplitudes in gauge theories*, *Nucl.Phys.* **B379** (1992) 451–561.
- [3] W. Siegel, *Supersymmetric Dimensional Regularization via Dimensional Reduction*, *Phys.Lett.* **B84** (1979) 193.
- [4] E. N. Glover, C. Oleari, and M. Tejeda-Yeomans, *Two loop QCD corrections to gluon-gluon scattering*, *Nucl.Phys.* **B605** (2001) 467–485, [[hep-ph/0102201](#)].

- [5] C. Anastasiou, E. N. Glover, C. Oleari, and M. Tejeda-Yeomans, *Two loop QCD corrections to massless quark gluon scattering*, *Nucl.Phys.* **B605** (2001) 486–516, [[hep-ph/0101304](#)].
- [6] Z. Bern, A. De Freitas, and L. J. Dixon, *Two loop helicity amplitudes for gluon-gluon scattering in QCD and supersymmetric Yang-Mills theory*, *JHEP* **0203** (2002) 018, [[hep-ph/0201161](#)].
- [7] Z. Bern, A. De Freitas, and L. J. Dixon, *Two loop helicity amplitudes for quark gluon scattering in QCD and gluino gluon scattering in supersymmetric Yang-Mills theory*, *JHEP* **0306** (2003) 028, [[hep-ph/0304168](#)].
- [8] W. B. Kilgore, *Regularization Schemes and Higher Order Corrections*, *Phys.Rev.* **D83** (2011) 114005, [[arXiv:1102.5353](#)].
- [9] R. Boughezal, K. Melnikov, and F. Petriello, *The four-dimensional helicity scheme and dimensional reconstruction*, *Phys.Rev.* **D84** (2011) 034044, [[arXiv:1106.5520](#)].
- [10] I. Jack, D. Jones, and K. Roberts, *Equivalence of dimensional reduction and dimensional regularization*, *Z.Phys.* **C63** (1994) 151–160, [[hep-ph/9401349](#)].
- [11] I. Jack, D. Jones, and K. Roberts, *Dimensional reduction in nonsupersymmetric theories*, *Z.Phys.* **C62** (1994) 161–166, [[hep-ph/9310301](#)].
- [12] R. Harlander, P. Kant, L. Mihaila, and M. Steinhauser, *Dimensional Reduction applied to QCD at three loops*, *JHEP* **0609** (2006) 053, [[hep-ph/0607240](#)].
- [13] R. Harlander, D. Jones, P. Kant, L. Mihaila, and M. Steinhauser, *Four-loop beta function and mass anomalous dimension in dimensional reduction*, *JHEP* **0612** (2006) 024, [[hep-ph/0610206](#)].
- [14] R. Harlander, P. Kant, L. Mihaila, and M. Steinhauser, *Dimensional reduction applied to QCD at higher orders*, [[arXiv:0706.2982](#)].
- [15] A. Signer and D. Stöckinger, *Factorization and regularization by dimensional reduction*, *Phys.Lett.* **B626** (2005) 127–138, [[hep-ph/0508203](#)].
- [16] W. Beenakker, H. Kuijf, W. van Neerven, and J. Smith, *QCD Corrections to Heavy Quark Production in  $p$  anti- $p$  Collisions*, *Phys.Rev.* **D40** (1989) 54–82.
- [17] J. Smith and W. van Neerven, *The Difference between  $n$ -dimensional regularization and  $n$ -dimensional reduction in QCD*, *Eur.Phys.J.* **C40** (2005) 199–203, [[hep-ph/0411357](#)].
- [18] Z. Kunszt, A. Signer, and Z. Trocsanyi, *One loop helicity amplitudes for all  $2 \rightarrow 2$  processes in QCD and  $N=1$  supersymmetric Yang-Mills theory*, *Nucl.Phys.* **B411** (1994) 397–442, [[hep-ph/9305239](#)].
- [19] E. Gardi and L. Magnea, *Infrared singularities in QCD amplitudes*, *Nuovo Cim.* **C32N5-6** (2009) 137–157, [[arXiv:0908.3273](#)].
- [20] T. Becher and M. Neubert, *Infrared singularities of scattering amplitudes in perturbative QCD*, *Phys.Rev.Lett.* **102** (2009) 162001, [[arXiv:0901.0722](#)].
- [21] T. Becher and M. Neubert, *On the Structure of Infrared Singularities of Gauge-Theory Amplitudes*, *JHEP* **0906** (2009) 081, [[arXiv:0903.1126](#)].
- [22] W. B. Kilgore, *The Four Dimensional Helicity Scheme Beyond One Loop*, *Phys.Rev.* **D86** (2012) 014019, [[arXiv:1205.4015](#)].
- [23] G. F. Sborlini, D. de Florian, and G. Rodrigo, *Double collinear splitting amplitudes at next-to-leading order*, *JHEP* **1401** (2014) 018, [[arXiv:1310.6841](#)].

- [24] A. Signer and D. Stöckinger, *Using Dimensional Reduction for Hadronic Collisions*, *Nucl.Phys.* **B808** (2009) 88–120, [[arXiv:0807.4424](#)].
- [25] D. Stöckinger, *Regularization by dimensional reduction: consistency, quantum action principle, and supersymmetry*, *JHEP* **0503** (2005) 076, [[hep-ph/0503129](#)].
- [26] D. Capper, D. Jones, and P. van Nieuwenhuizen, *Regularization by Dimensional Reduction of Supersymmetric and Nonsupersymmetric Gauge Theories*, *Nucl.Phys.* **B167** (1980) 479.
- [27] S. Catani, *The Singular behavior of QCD amplitudes at two loop order*, *Phys.Lett.* **B427** (1998) 161–171, [[hep-ph/9802439](#)].
- [28] G. F. Sterman and M. E. Tejeda-Yeomans, *Multiloop amplitudes and resummation*, *Phys.Lett.* **B552** (2003) 48–56, [[hep-ph/0210130](#)].
- [29] M. E. Machacek and M. T. Vaughn, *Two Loop Renormalization Group Equations in a General Quantum Field Theory. 1. Wave Function Renormalization*, *Nucl.Phys.* **B222** (1983) 83.
- [30] M. E. Machacek and M. T. Vaughn, *Two Loop Renormalization Group Equations in a General Quantum Field Theory. 2. Yukawa Couplings*, *Nucl.Phys.* **B236** (1984) 221.
- [31] M. E. Machacek and M. T. Vaughn, *Two Loop Renormalization Group Equations in a General Quantum Field Theory. 3. Scalar Quartic Couplings*, *Nucl.Phys.* **B249** (1985) 70.
- [32] M.-x. Luo, H.-w. Wang, and Y. Xiao, *Two loop renormalization group equations in general gauge field theories*, *Phys.Rev.* **D67** (2003) 065019, [[hep-ph/0211440](#)].
- [33] R. J. Gonsalves, *Dimensionally Regularized Two Loop On-Shell Quark Form-factor*, *Phys.Rev.* **D28** (1983) 1542.
- [34] R. V. Harlander, *Virtual corrections to  $g g \rightarrow H$  to two loops in the heavy top limit*, *Phys.Lett.* **B492** (2000) 74–80, [[hep-ph/0007289](#)].
- [35] T. Gehrmann, E. Glover, T. Huber, N. Ikizlerli, and C. Studerus, *Calculation of the quark and gluon form factors to three loops in QCD*, *JHEP* **1006** (2010) 094, [[arXiv:1004.3653](#)].
- [36] S. Moch, J. Vermaseren, and A. Vogt, *The Quark form-factor at higher orders*, *JHEP* **0508** (2005) 049, [[hep-ph/0507039](#)].
- [37] S. Moch, J. Vermaseren, and A. Vogt, *Three-loop results for quark and gluon form-factors*, *Phys.Lett.* **B625** (2005) 245–252, [[hep-ph/0508055](#)].
- [38] T. Becher, M. Neubert, and B. D. Pecjak, *Factorization and Momentum-Space Resummation in Deep-Inelastic Scattering*, *JHEP* **0701** (2007) 076, [[hep-ph/0607228](#)].
- [39] S. Moch, J. Vermaseren, and A. Vogt, *The Three loop splitting functions in QCD: The Nonsinglet case*, *Nucl.Phys.* **B688** (2004) 101–134, [[hep-ph/0403192](#)].
- [40] T. Hahn, *Generating Feynman diagrams and amplitudes with FeynArts 3*, *Comput.Phys.Comm.* **140** (2001) 418–431, [[hep-ph/0012260](#)].
- [41] M. Jamin and M. E. Lautenbacher, *TRACER: Version 1.1: A Mathematica package for gamma algebra in arbitrary dimensions*, *Comput.Phys.Comm.* **74** (1993) 265–288.
- [42] S. Laporta, *High precision calculation of multiloop Feynman integrals by difference equations*, *Int.J.Mod.Phys.* **A15** (2000) 5087–5159, [[hep-ph/0102033](#)].
- [43] A. Smirnov, *Algorithm FIRE – Feynman Integral REduction*, *JHEP* **0810** (2008) 107, [[arXiv:0807.3243](#)].

- [44] A. Smirnov and M. Tentyukov, *Feynman Integral Evaluation by a Sector decomposition Approach (FIESTA)*, *Comput.Phys.Commun.* **180** (2009) 735–746, [[arXiv:0807.4129](#)].
- [45] V. Spiridonov, *Anomalous Dimension of  $g^{*2}$  (Muon-neutrino) and Beta Function*, .
- [46] K. Chetyrkin, B. A. Kniehl, and M. Steinhauser, *Three loop  $O(\alpha_s^{*2} G(F) M(t)^{*2})$  corrections to hadronic Higgs decays*, *Nucl.Phys.* **B490** (1997) 19–39, [[hep-ph/9701277](#)].
- [47] K. Chetyrkin, B. A. Kniehl, and M. Steinhauser, *Decoupling relations to  $O(\alpha_s^{*3})$  and their connection to low-energy theorems*, *Nucl.Phys.* **B510** (1998) 61–87, [[hep-ph/9708255](#)].